Year 12 Mathematics

IAS 2.1

Coordinate Geometry

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NCEA 2 Internal Achievement Standard 2.1 – Coordinate Geometry

This achievement standard involves applying coordinate geometry methods in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply coordinate geometry methods in solving problems.	• Apply coordinate geometry methods, using relational thinking, in solving problems.	• Apply coordinate geometry methods, using extended abstract thinking, in solving problems.

◆ This achievement standard is derived from Level 7 of The New Zealand Curriculum and is related to the achievement objective:
- ❖ apply coordinate geometry techniques to points and lines.

◆ Apply coordinate geometry methods in solving problems involves:
- ❖ selecting and using methods
- ❖ demonstrating knowledge of geometric concepts and terms
- ❖ communicating using appropriate representations.

◆ Relational thinking, involves one or more of:
- ❖ selecting and carrying out a logical sequence of steps
- ❖ connecting different concepts or representations
- ❖ demonstrating understanding of concepts
- ❖ forming and using a model;

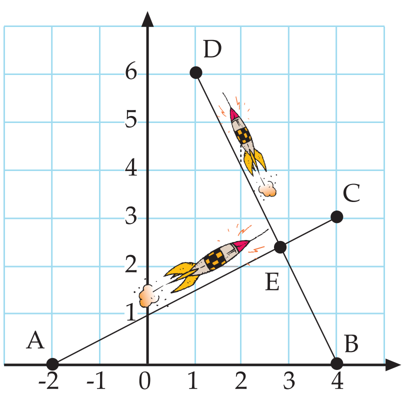
and also relating findings to a context, or communicating thinking using appropriate mathematical statements.

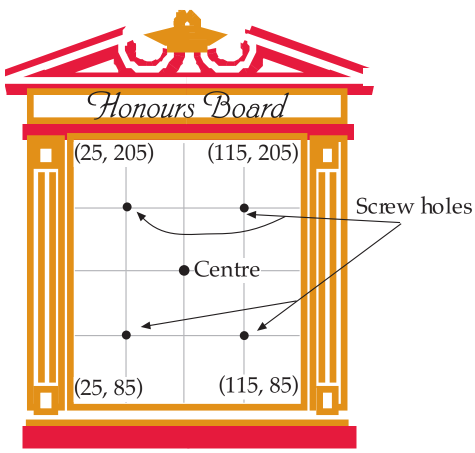
◆ Extended abstract thinking involves one or more of:
- ❖ devising a strategy to investigate or solve a problem
- ❖ identifying relevant concepts in context
- ❖ developing a chain of logical reasoning, or proof
- ❖ forming a generalisation;

and also using correct mathematical statements, or communicating mathematical insight.

◆ Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.

◆ Methods include a selection from those related to:
- ❖ distance between points
- ❖ midpoints
- ❖ the gradient of a line
- ❖ the equation of a line
- ❖ parallel, perpendicular and intersecting lines.





Cartesian Coordinates

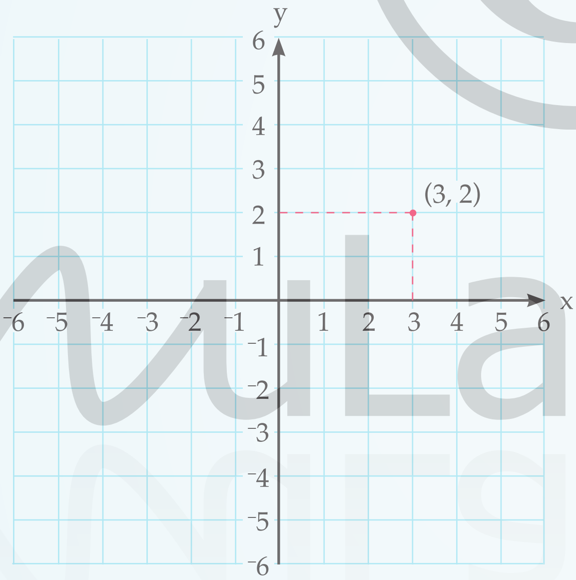
Cartesian Coordinates

Cartesian coordinates are a pair of numerical coordinates that uniquely specify a point in a plane. A plane is a flat two dimensional surface like a sheet of paper.

The construction of two perpendicular axes on a plane is called a Cartesian plane after René Descartes (1596 – 1650).

The two axes are usually denoted by x and y. The x axis is the horizontal axis and the y axis the vertical axis. The point where the axes cross is called the origin, i.e. where x = 0 and y = 0.

Cartesian coordinates are often called rectangular coordinates. To understand why see the dotted rectangle in the diagram below.

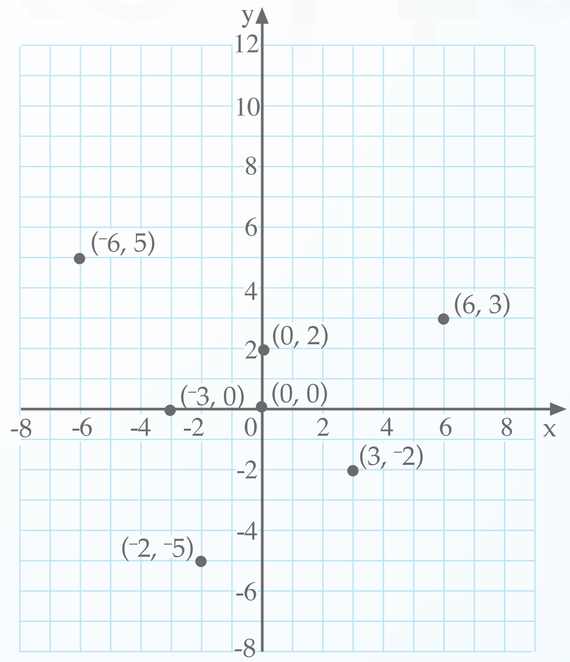


We represent a pair of Cartesian coordinates in the form (x, y), often called ordered pair notation.

Every time you graph an equation or plot a point on a Cartesian plane, you are using the work of René Descartes. Descartes, a French mathematician and philosopher, was born in La Haye, France (now named in his honour) on March 31, 1596. He died in 1650. (Source: Encyclopedia.com).

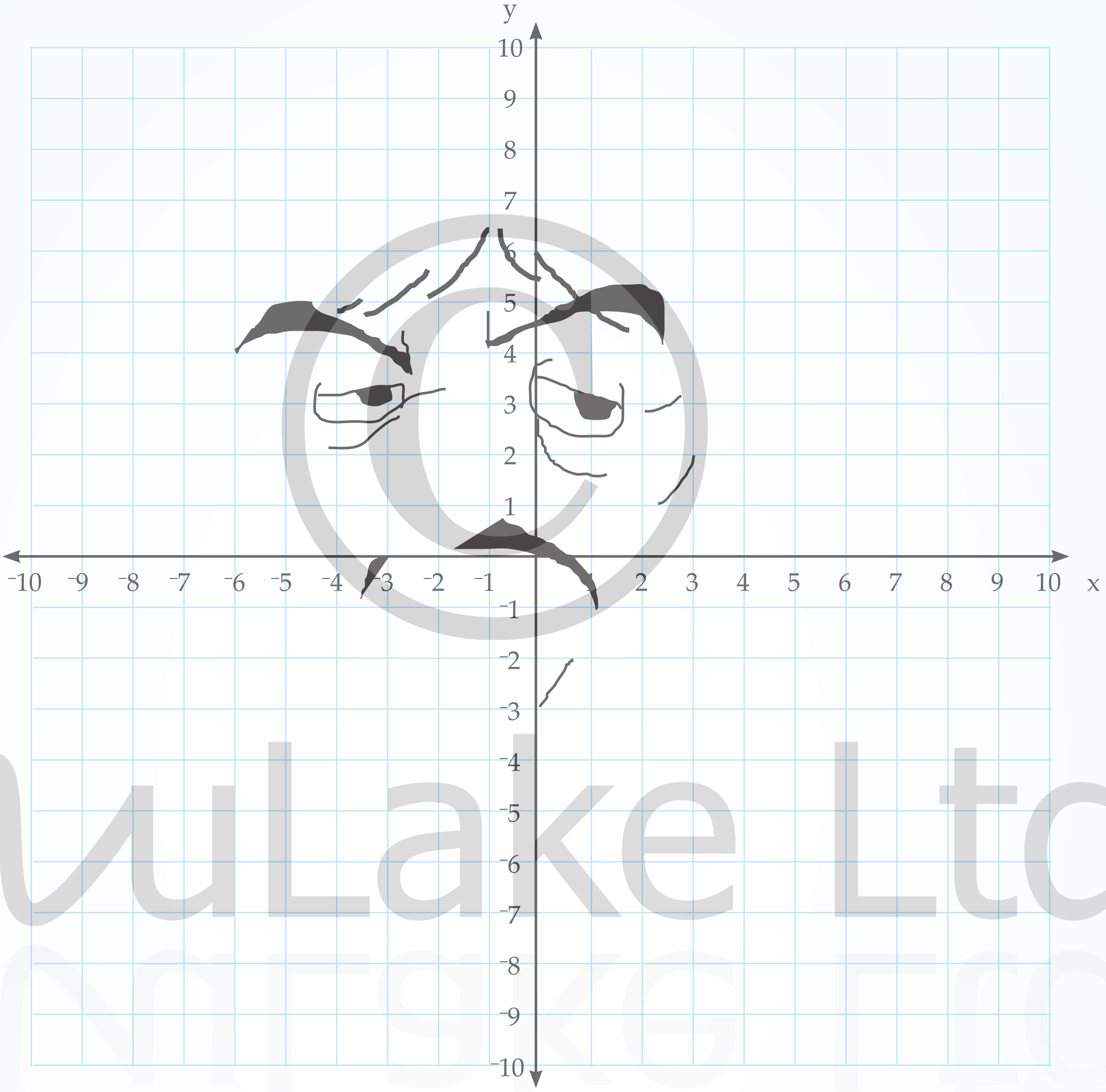
Extra Notes

To plot points on a Cartesian plane we always go across first (x axis) and then up or down (y axis). A good way to remember this is that an aircraft always goes along the runway before it goes up in the air. Points on a Cartesian plane are written (x, y).



Achievement – Complete the caricature of René Descartes by plotting the coordinates on the grid below.

2.



Plot $(^-1, 1.5)$ to $(^-0.5, 1)$ to $(^-2.5, ^-0.5)$ to $(^-3, 0)$ to $(^-2.5, 3.5)$.

Lift you pen and plot $(^-5, 4)$ to $(^-5, 3)$ to $(^-5.5, 2)$ to $(^-4, 0)$ to $(^-4, ^-3)$ to $(^-5, ^-6)$ to $(^-2, ^-3.25)$ to $(0, ^-7)$ to $(2, ^-5)$ to $(4, ^-2)$ to $(2.5, ^-1)$ to $(3, 2)$ to $(3.5, 4.5)$ to $(2.5, 4.5)$.

Lift you pen and plot $(2.5, ^-1)$ to $(2, ^-2)$ to $(0, ^-3)$ to $(^-0.5, ^-3.5)$ to $(^-4, ^-3)$.

Lift you pen and plot $(^-0.5, ^-0.5)$ to $(^-2.5, ^-1.5)$ to $(^-2.5, ^-0.5)$ to $(^-0.5, ^-0.5)$

Lift your pen and plot $(5, ^-7.5)$ to $(5, ^-6.5)$ to $(8, ^-5)$ to $(9.5, ^-3)$ to $(9, 0)$ to $(6, 1)$ to $(4.5, 3)$ to $(5, 6)$ to $(4, 8)$ to $(3, 9)$ to $(0, 8)$ to $(^-1.5, 8.5)$ to $(^-3, 8)$ to $(^-5, 6)$ to $(^-5, 5)$ to $(^-6, 4)$ to $(^-7, 1)$ to $(^-8, 0)$ to $(^-6, ^-3)$ to $(^-5, ^-4)$ to $(^-4.5, ^-4.5)$

Lift your pen and plot $(^-3, ^-4.25)$ to $(^-2, ^-5.5)$ to $(^-3, ^-7.5)$.

Gradient of a Straight Line

Gradient of a Straight Line

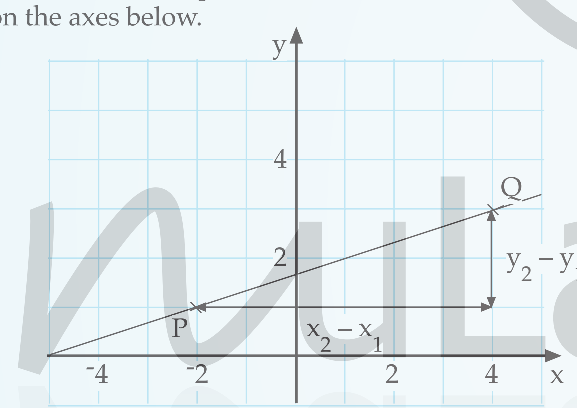
The gradient of a line is a measure of how steep a line is. We describe a line as having a positive gradient (slope) if it is drawn from the bottom left to the top right on the Cartesian plane. It is defined as having a negative gradient (slope) if it is drawn from the bottom right to the top left on the Cartesian plane. A horizontal line has a gradient of zero.

In Year 11, we defined the gradient of a line as

$$\text{Gradient} = \frac{\text{distance up (down)}}{\text{distance along}}$$

This year we look to formalise this by using a formula with similar notation to that used in the previous two sections.

Consider the two points P(⁻2, 1) and Q(4, 3) drawn on the axes below.



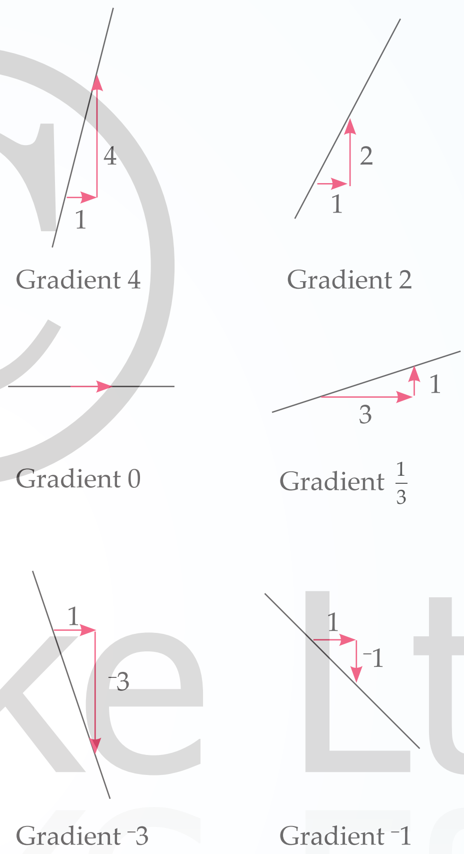
If we represent the coordinates of the point P as (x_1, y_1) and the coordinates of Q as (x_2, y_2), then we can find the gradient (slope) of the line PQ by using the formula.

$$\text{Gradient (m)} \quad = \frac{y_2 - y_1}{x_2 - x_1}$$

Effectively we are finding the difference in the vertical values of the two points (i.e. $y_2 - y_1$) and dividing it by the difference in the horizontal values of the two points (i.e. $x_2 - x_1$).

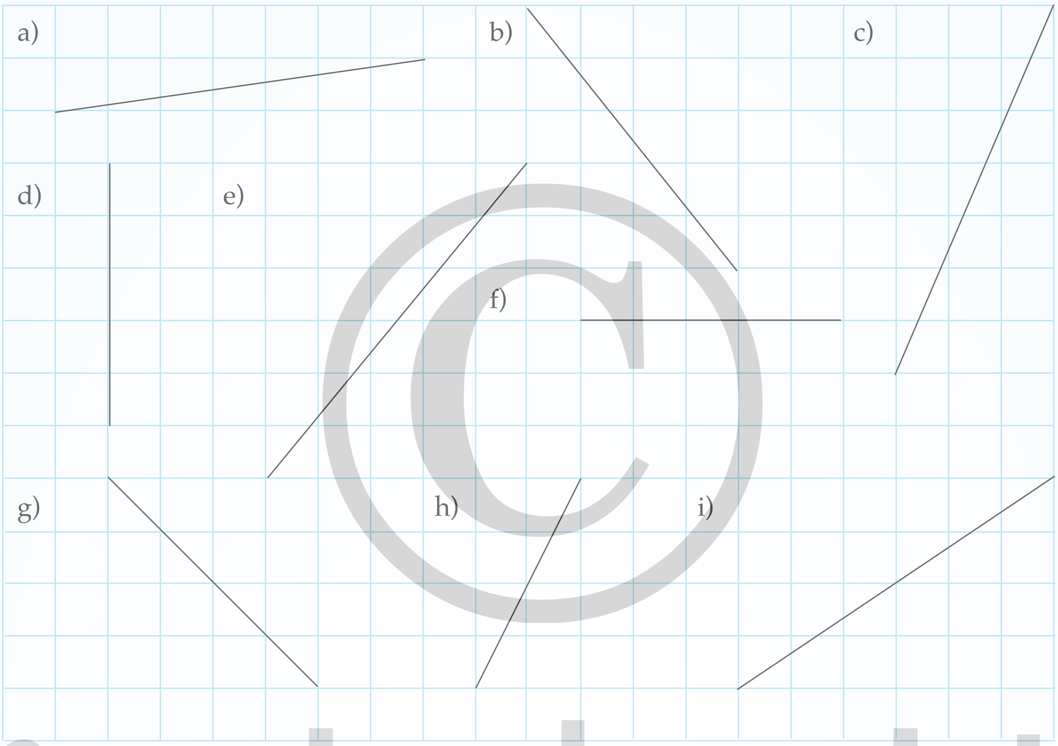
This is the same as calculating the distance along, and then up or down, from one point to another as we did in Year 11.

Extra Notes



Achievement – Answer the following questions.

63. Calculate the gradients of the line segments drawn below.



64. Draw line segments on the grid below with the following gradients.

a) Gradient = $^{-}6$ b) Gradient = 8 c) Gradient = $\frac{^{-}2}{7}$ d) Gradient = $\frac{1}{4}$

e) Gradient = $\frac{5}{8}$ f) Gradient = $^{-}1\frac{2}{3}$ g) Gradient = 2.5 h) Gradient = $^{-}2.333$

Achievement – Answer the following questions.

126. Are the lines A and B passing through the given pairs of points, parallel, perpendicular or neither?

a) A: $(3, 4), (5, 3)$
B: $(^-1, ^-2), (1, ^-1)$

b) A: $(1, 4), (4, 2)$
B: $(0, 5), (^-6, ^-4)$

c) A: $(1, ^-2), (4, ^-8)$
B: $(2, 2), (1, 4)$

d) A: $(1, 1), (3, 1)$
B: $(6, ^-4), (^-9, ^-4)$

e) A: $(6, ^-2), (3, ^-3)$
B: $(0, 1), (4, ^-11)$

f) A: $(4, 3), (^-2, 5)$
B: $(5, 8), (4, 11)$

127. By identifying the gradient of each line determine whether the graphs of the pair of lines would be parallel, perpendicular or neither.

a) $y = 3x - 4$
$y = 3x + 1$

b) $y = 2x - 1$
$y = {}^-0.5x + 5$

c) $y = {}^-4x + 1$
$4y = x + 3$

d) $y = 2x + 1$
$y = \frac{1}{2}x - 2$

e) $y = 2$
$3y = 9$

f) $2y = {}^-4x + 3$
$2x + y = {}^-5$

g) $3x - 5y = 9$
$3y = {}^-5x + 6$

h) $y = {}^-7x + 2$
$x + 7y = 8$

The Solution of Simultaneous Equations Using Algebra

The Elimination Method

There are a number of different methods for solving simultaneous equations. Two of these include substitution and elimination.

The single requirement to use elimination is that both equations should be in the same form, such as

$$2x + 3y = 5$$
$$x - 6y = 8$$

To be in the same form, both equations should have their x, y, equal signs and constant terms in the same relative position.

If the equations are not in the same form, then we have to manipulate them prior to starting.

For example

$$y = 3x + 5$$

and $2y - x = 5$

would be rearranged as

$$y - 3x = 5$$

and $2y - x = 5$

The method of elimination involves multiplying one or both of the equations by a constant so that we are able to ADD one equation to the other in order to eliminate one variable. In selecting the constant to multiply by, we are attempting to get the coefficient of x (or y) the same but of opposite sign. Multiplying the first equation above by negative 2 would give

$$^{-}2y + 6x = {}^{-}10$$

and $2y - x = 5$

Adding these two equation vertically gives

$$5x = {}^{-}5$$

The resulting equation with one variable is then easily solved.

$$x = {}^{-}1$$

This result is substituted into either of the first two equations to get the solution of $y = 2$.

If we always make sure the coefficients are of opposite signs, then we can ADD the equation. This removes a source of error when students subtract equations and have difficulty with subtracting negative coefficients.

After solving an equation always check your answer by substituting it back into the equation.

Extra Notes

Example

Solve the simultaneous equations.

$$6x + 4y = 6$$
$$4y = 2x + 14$$

Move the 2x to the left side so the equations are in the same form. We do this by subtracting 2x from both sides.

$$6x + 4y = 6$$
$$^{-}2x + 4y = 14$$

Assign each equation a label

$$6x + 4y = 6 \qquad (1)$$
$$^{-}2x + 4y = 14 \qquad (2)$$

Multiply equation (2) by 3 so the coefficients of x are 6 and $^{-}6$.

$$6x + 4y = 6 \qquad (1)$$

3 x (2) $\quad ^{-}6x + 12y = 42 \qquad (3)$

Now combine both equations by adding down

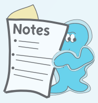
(1) + (3) $\quad 0 + 16y = 48$

$$y = 3 \qquad (4)$$

Substituting back $6x + 4 \times 3 = 6$

$$x = {}^{-}1$$

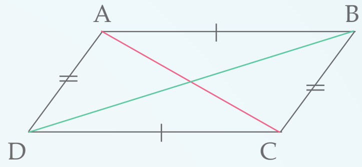
The solution to the equations is ($^{-}1$, 3)



Quadrilaterals and their Properties

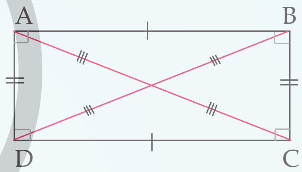
To answer some of the questions in this Achievement Standard it is necessary to be familiar with the different types of quadrilaterals and their properties. These are summarised below.

Parallelogram



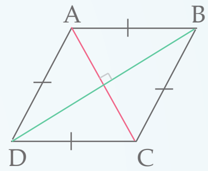
- Opposite sides are equal (congruent) (AB = DC, AD = BC).
- Opposite sides are parallel (AB // DC, AD // BC).
- Opposite angles are equal (congruent) ($\angle A = \angle C$, $\angle B = \angle D$)
- The diagonals of a parallelogram bisect each other.
- Each diagonal of a parallelogram separates the parallelogram into two equal congruent triangles.
- Any pair of consecutive angles are supplementary (add to 180°).

Rectangle



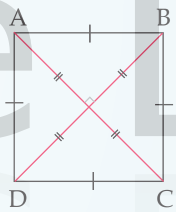
- All the properties of a parallelogram apply.
- The diagonals of a rectangle are equal (congruent).
- All angles are right-angles.

Rhombus



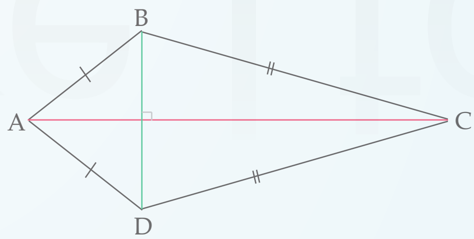
- All the properties of a parallelogram apply.
- All sides are equal (congruent).
- The diagonals bisect the angles.
- The diagonals are perpendicular bisectors of each other.
- The diagonals divide the rhombus into four equal (congruent) right-angled triangles.

Square



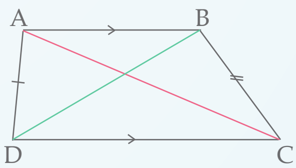
- All the properties of a rectangle apply.
- All the properties of a rhombus apply.
- The diagonals form four isosceles triangles.

Kite



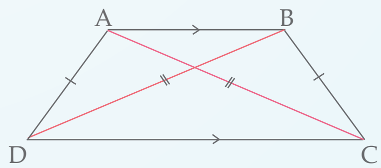
- Two pairs of sides are equal (congruent) (BC = DC, AB = AD).
- The diagonals are perpendicular.
- One pair of opposite angles are equal (congruent) ($\angle B = \angle D$).
- One of the diagonals (AC) bisects a pair of opposite angles.

Trapezium



- One pair of opposite sides is parallel.
- Each lower base angle is supplementary (adds to 180°) to the upper base angle on the same side.

Isosceles Trapezium



- All the properties of a trapezium apply.
- The diagonals are equal (congruent).
- The non-parallel sides are equal (congruent) (AD = BC).
- The angles on either side of the bases are equal (congruent) ($\angle A = \angle B$, $\angle C = \angle D$).

Page 16

63. a) $\frac{1}{7}$

b) $\frac{^{-}5}{4}$

c) $\frac{7}{3}$

d) undefined

e) $\frac{6}{5}$

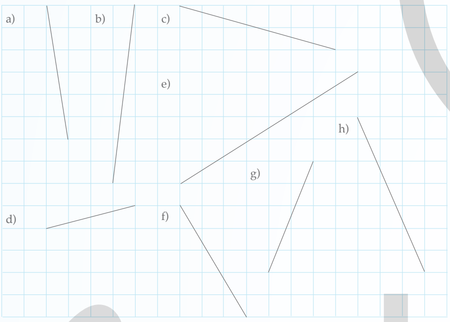
f) 0

g) $^{-}1$

h) 2

i) $\frac{2}{3}$

64. a), b), c), d), e), f), g), h)



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65. $\frac{1}{3}$ (0.33)

66. $\frac{9}{2}$ (4.5)

67. $^{-}2$

68. $\frac{1}{3}$ (0.33)

69. 1

70. $\frac{9}{20}$ (0.45)

71. $\frac{1}{5}$ (0.2)

72. 2.0625

73. 3

74. $\frac{b}{a}$

75. $\frac{^{-}b}{4a}$

76. $^{-}0.44$ (2 sf)

77. $^{-}0.034$ (2 sf)

78. 2

79. $\frac{7a+1}{a+4}$

Page 17 cont...

80. $b = {^{-}5}$

81. Gradient AB $= \frac{-2}{3}$ ($^{-}0.67$)

Gradient BC $= \frac{-2}{3}$ ($^{-}0.67$)

Gradient AC $= \frac{-2}{3}$ ($^{-}0.67$)

Since AB, BC and AC all have the same gradient they must lie on the same line hence they are collinear.

82. $a = \frac{^{-}26}{5}$ ($^{-}5.2$)

83. $a = {^{-}8}$

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84. $k = {^{-}9}$

85. $k = 11.25$

86. $k = \frac{23}{3}$ (7.67)

87. $a = 3, b = 3$

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88. $y = \frac{2}{3}x + \frac{7}{3}$ or $2x - 3y + 7 = 0$

89. $y = \frac{^{-}4}{5}x - \frac{6}{5}$ or $4x + 5y + 6 = 0$

90. $y = \frac{1}{7}x + \frac{23}{7}$ or $x - 7y + 23 = 0$

91. $y = x - 6$ or $x - y - 6 = 0$

92. $y = 5$ or $y - 5 = 0$

93. $y = \frac{^{-}1}{2}x + \frac{3}{2}$ or $x + 2y - 3 = 0$

94. $y = 3x - 13$ or $3x - y - 13 = 0$

95. $y = \frac{1}{2}x + 8$ or $x - 2y + 16 = 0$

96. $y = \frac{^{-}2}{5}x - \frac{33}{5}$
or $2x + 5y + 33 = 0$

97. $y = \frac{3}{2}x - \frac{11}{2}$ or $3x - 2y - 11 = 0$

98. $y = \frac{^{-}8}{3}x - \frac{7}{3}$ or $8x + 3y + 7 = 0$

99. $y = {^{-}1.5}x + 6.65$
or $3x + 2y - 13.3 = 0$

100. $y = 2x - 5$ or $2x - y - 5 = 0$

101. $y = {^{-}2}x + 1$ or $2x + y - 1 = 0$

102. $y = 3x + 10$ or $3x - y + 10 = 0$

103. $y = \frac{1}{2}x - 7$ or $x - 2y - 14 = 0$

Page 20 cont...

104. $y = \frac{1}{6}x + \frac{8}{3}a$ or $x - 6y + 16a = 0$

105. $y = {^{-}q}$ or $y + q = 0$

106. $y = \frac{^{-}5}{2}x + 10.4$
or $5x + 2y - 20.8 = 0$

107. $y = 3x - 2$ or $3x - y - 2 = 0$

Page 21

108. $m = \frac{-7}{2}$ ($^{-}3.5$)

y intercept $= \frac{9}{2}$ (4.5)

109. $m = \frac{-4}{7}$ ($^{-}0.57$)

y intercept $= \frac{13}{7}$ (1.86)

110. $m = \frac{-1}{3}$ ($^{-}0.33$)

y intercept $= \frac{3}{4}$ (0.75)

111. $m = \frac{^{-}4}{5}$ ($^{-}0.8$)

y intercept $= \frac{6}{5}$ (1.2)

112. $y = \frac{2}{3}x - \frac{5}{3}$ or $2x - 3y - 5 = 0$

113. $y = 8x - 12$ or $8x - y - 12 = 0$

114. $y = \frac{17}{4}x + \frac{1}{4}$ or $17x - 4y + 1 = 0$

115. AB, $y = \frac{3}{5}x + \frac{34}{5}$
or $3x - 5y + 34 = 0$
BC, $y = 12x - 16$
or $12x - y - 16 = 0$

116. $y = 3x + 13$ or $3x - y + 13 = 0$

117. $y = \frac{3}{4}x - \frac{21}{4}$ or $3x - 4y - 21 = 0$

118. The points B, D and E.

119. $y = \frac{1}{2}x + 2$ or $x - 2y + 4 = 0$

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120. $6x + 5y = 0$

121. Equation $y = 2x + 13$
or $2x - y + 13 = 0$
Points A, B and C can be substituted into this equation.

122. a) $y = \frac{^{-}1}{4}x + 2$ or $x + 4y - 8 = 0$

b) $y = \frac{1}{2}x + 1$ or $x - 2y + 2 = 0$